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An Integral Solution for the Blasius Equation

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Abstract

The current paper is aimed to propose an approximate analytical method for solving the well-known Blasius boundary-layer problem by combining the Green's function method and the best approximation theorem. The Blasius equation is the nonlinear ordinary differential equation for the laminar fluid flow over a sheet. The proposed integral solution is developed via the use of the Green's function idea as well as approximating the nonlinear term of the Blasius Equation. Specifically, the novelty of the present paper originates from proposing an innovative approximation for the nonlinear term of the Blasius problem by using a trigonometric expansion. Results reveal that the proposed integral solution coupled with the trigonometric approximation for the nonlinear term leads to a nearly accurate solution which is in agreement with the numerical results.

1. Introduction

At the first of the last century, fundamentals of the boundary-layer theory is established by Prandtl [1, 2]. According to this theory, the two-dimensional steady-state laminar incompressible flow of a fluid with constant velocity on a semi-infinite flat plate is stated as [2, 3]

$$\frac{\partial v_1}{\partial x_1}+\frac{\partial v_2}{\partial x_2}=0 \tag{1}$$

$$v_1\frac{\partial v_1}{\partial x_1}+v_2\frac{\partial v_1}{\partial x_2}=\upsilon\frac{\partial^2 v_1}{\partial x_2^2} \tag{2}$$

with the following boundary conditions

$$v_1(x_1,0)=v_2(x_1,0)=0,\quad v_1(x_1,0)=u_\infty,\qquad \frac{\partial v_1}{\partial x_2}(x_1,x_\infty)=0 \tag{3}$$

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Here, v_1 and v_2 are the velocity components in x_1 and x_2 directions, respectively. u_∞ is the velocity of the free stream and υ is the kinematic viscosity of the fluid.

In 1908, Blasius [4] proposed the similarity solution of Eq. (2) by introducing the following similarity transformation

$$x = x_2\sqrt{\frac{u_\infty}{\upsilon x_1}},\;\; v_1 = u_\infty f'(x),\;\; v_2 = \frac{1}{2}\sqrt{\frac{u_\infty \upsilon}{x_1}}(xf'(x) - f(x)) \tag{4}$$

which reduces the boundary-layer governing equations and the associated boundary conditions to

$$\frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} = 0 \tag{5}$$

$$v_1\frac{\partial v_1}{\partial x_1} + v_2\frac{\partial v_1}{\partial x_2} = \upsilon\frac{\partial^2 v_1}{\partial x_2^2} \tag{6}$$

in which x_∞ stands for the boundary-layer thickness. Eq. (5) indicates the classical Blasius boundary-layer problem which is well-known as the origin of the use of the similarity technique in the fluid boundary-layer Eqs. [5]. Since 1908, several different but relevant branches of problems have been developed based on Blasius's method such as Refs. [6-9]. Regardless of the fact that the Blasius problem is an old one, no closed-form solution has been proposed for it by now [10, 11]. However, several essays have been published dealing with the numerical or analytical solutions of this problem. Several works can be found available in the literature dealing with the solution of the Blasius problem by various numerical methods. Regarding to the fact that numerical approaches are out of the scope of the current study, readers are referred to the review paper of Boyd [12].

So far, several analytical methods have been employed for solving the well-known Blasius problem. Blasius [4] himself solved the problem by using a power series expansions method. Yu and Chen [13] used the differential transformation approach to solve the problem. Lin [14] approximated the Blasius problem by developing the parameter iteration method. Liao [15] analytically solved the Blasius equation using the Homotopy Analysis Method (HAM). Furthermore, HAM is employed in the works of authors [16, 17] to investigate the Blasius equation. Wang [18] applied the Adomian Decomposition Method (ADM) and presented an analytical solution for the Blasius equation. Wazwaz [19] used the modified ADM to obtain analytical solution of Blasius equation. Also, in another study, Wazwaz [20] employed the Variational Iteration Method (VIM) to solve Blasius equation analytically. In both essays, Wazwaz combined the solution method with the diagonal Padé approximation to handle the boundary condition at infinity. Abbasbandy [21] employed ADM to solve the Blasius equation. Besides, a comparison was reported with the results of the Homotopy Perturbation Method (HPM). Parand et al. [5] applied the Sinc-collocation method [22] and obtained an approximate solution for Blasius equation converging at an exponential rate. Also Parand and Taghavi [23] proposed an algorithm to solve Blasius equation by using the collocation and tau methods based on the rational scaled generalized Laguerre. Later, Parand et al. [24] presented an analytical solution for the Blasius Equation employing the collocation and the rational Gegenbauer functions.

Yun [25] and Savaş [26] proposed hyperbolic solutions for the Blasius problem. The approximate analytical solution of Yun [25] was in the form of the logarithm of the hyperbolic cosine function, while Savaş [26] presented a two-parameter hyperbolic tangent function. An innovative algorithm named Optimal Homotopy Asymptotic Method (OHAM) is proposed by Marinca and Herişanu [11] to get an explicit analytical solution of the Blasius problem. Xu and Guo [10] reported a semi-analytical solution of Blasius equation by proposing the Fixed-Point Method (FPM). He [27], who introduced the basic idea of the homotopy perturbation method (HPM) for the first time, released an article and solved the Blasius Equation using the variational iteration method [27]. Later, He [28] coupled an iteration technique with the perturbation method and developed an analytical approximation for the Blasius equation. According to this background, analytical solutions proposed for the Blasius equation are commonly based on homotopy methods, variational methods, perturbation methods and the collocation methods. Solutions originates from the collocation method are generally based on approximating the solution using a polynomial which satisfies boundary conditions.

In the current paper, an analytical approach is proposed to evaluate the approximate solution of the Blasius boundary value problem. The developed method consists of splitting the nonlinear equation into two linear and nonlinear parts, applying Green’s function method and approximating the nonlinear term by the series expansion. Basically, the innovation of the proposed solution is due to using a trigonometric expansion to specifically approximating the nonlinear term of the Blasius problem.

2. Integral Solution

The method of the solution of the non-homogenous differential equations via the use of the Green’s functions as an integral type is a fundamental concept in functional analysis [29]. The present analytical approach is mainly based on the idea of the integral solution as well as the Green’s function. First, the Blasius equation expressed via Eq. (5), can be recast to

$$f''' = -\frac{1}{2}ff'' = F(x,f,f'') \tag{7}$$

where $F(x,f,f'')$ is considered as an unknown function which includes the nonlinear term of the Blasius Equation. According to the integral solution idea, a fundamental solution should exist for Eq. (7) via the use of the Green's function [29]. On this basis, the solution of Eq. (7) can be written as in the following form

$$f(x) = \int_0^{x_\infty} G(x,s)F(s,f,f'')ds \tag{8}$$

where $G(x,x')$ is the Green's function which can be obtained from

$$-\frac{d^3}{dx^3}[G(x,x')] = \delta(x-x') \;\; \rightarrow \;\; \begin{cases} \dfrac{d^3}{dx^3}[G(x,x')] = 0 \quad 0 \le x \le x' \\ \dfrac{d^3}{dx^3}[G(x,x')] = 0 \quad x' \le x \le x_\infty \end{cases} \tag{9}$$

where $-d^3/dx^3$ is the adjoint operator of the L. Also, x' is a variable between *0* and x_∞. Green's function is simply achieved as

$$\begin{cases} G(x,x') = a\dfrac{x^2}{2} + bx + c \quad 0 \le x \le x' \le x_\infty \\ G(x,x') = a'\frac{x^2}{2} + b'x + c' \quad 0 \le x' \le x \le x_\infty \end{cases} \tag{10}$$

Here, a, a', b, b', c and c' are arbitrary constants and should be obtained by imposing the following boundary condition

$$\begin{gathered} G(0,x')=G'(0,x')=0, \;\; G(x_\infty,x')=1 \\ G(x'_+,x')=G(x'_-,x') \\ G'(x'_+,x')=G'(x'_-,x') \\ G''(x'_+,x')-G''(x'_-,x')=1 \end{gathered} \tag{11}$$

Determining the constants of Eq. (10), the Green's function reads as

$$G(x,x') = \begin{cases} \left(1+\frac{1-x'}{x_\infty}\right)\frac{x^2}{2} \quad 0 \le x \le x' \le x_\infty \\ \left(\frac{1-x'}{x_\infty}\right)\frac{x^2}{2} + xx' - \frac{x'^2}{2} \quad 0 \le x' \le x \le x_\infty \end{cases} \tag{12}$$

Substituting the Green's function from Eq. (12) into Eq. (8), yields the following expression for f(x)

$$f(x) = \int_0^x \left[\left(\frac{1-s}{x_\infty}\right)\frac{x^2}{2} + sx - \frac{s^2}{2}\right] F(s,f,f'')ds + \int_x^{x_\infty} \left[\left(1 + \frac{1-s}{x_\infty}\right)\frac{x^2}{2}\right] F(s,f,f'')ds \quad (13)$$

In the next step, in order to find the solution of f(x), it is necessary to appropriately approximate the function $F(s,f,f'') = -1/2\, ff'$. Two various types of series expansions such as the Legendre's polynomial expansion and the trigonometric expansion are considered to approximate this unknown function.

2.1. The Legendre's Polynomial

In this section the function $F(s,f,f'')$ is approximated by a first degree polynomial in the form of $c_1x + c_2$ which can be considered as a Legendre polynomial. Substituting this linear expansion into the Eq. (13), results in the following

$$f(x) = \int_0^x \left[\left(\frac{1-s}{x_\infty}\right)\frac{x^2}{2} + sx - \frac{s^2}{2}\right] (c_1 s + c_2)ds + \int_x^{x_\infty} \left[\left(1 + \frac{1-s}{x_\infty}\right)\frac{x^2}{2}\right] (c_1 s + c_2)ds = \left[c_1\left(\frac{x_\infty^2+3x_\infty}{12}\right) + c_2\left(\frac{x_\infty+2}{4}\right)\right]x^2 - \left(\frac{c_2}{6}\right)x^3 - \left(\frac{c_1}{24}\right)x^4 \quad (14)$$

which should satisfy the following boundary conditions

$$f(0) = 0,\ \ f'(0) = 0 \quad (15a)$$

$$f'(x_\infty) = 1,\ \ f''(x_\infty) = 0 \quad (15b)$$

Here, the boundary conditions of Eq. (15a) are automatically satisfied. Thus, c_1 and c_2 are obtained using the two other boundary conditions which yields the following for f(x)

$$f(x) = \left[(-0.0254)\left(\frac{x_\infty^2 + 3x_\infty}{12}\right) + (0.0988)\left(\left(\frac{x_\infty + 2}{4}\right)\right)\right]x^2 + \left(\frac{0.0988}{6}\right)x^3 + \left(\frac{0.0254}{24}\right)x^4 \quad (16)$$

2.2. The Trigonometric Expansion

In this section, the function $F(s,f,f'')$ is approximated by a first degree polynomial within the form of $c_1 \mathrm{Sin}\left(\frac{\pi x}{2x_\infty}\right) + c_2 \mathrm{Cos}\left(\frac{\pi x}{2x_\infty}\right)$ which is a trigonometric series expansion. Substituting it into the Eq. (13), results in the following expression

$$f(x) = \int_0^x \left[\left(\frac{1-s}{x_\infty}\right)\frac{x^2}{2} + sx - \frac{s^2}{2}\right] \left[c_1 Sin\left(\frac{\pi x}{2x_\infty}\right) + c_2 Cos\left(\frac{\pi x}{2x_\infty}\right)\right]ds + \int_x^{x_\infty} \left[\left(1 + \frac{1-s}{x_\infty}\right)\frac{x^2}{2}\right] \left[c_1 Sin\left(\frac{\pi x}{2x_\infty}\right) + c_2 Cos\left(\frac{\pi x}{2x_\infty}\right)\right]ds \quad (17)$$

which leads to the following function for the f(x)

$$f(x) = c_1 \left\{ \left[-\frac{Cos\left(\frac{\pi x}{2x_\infty}\right)}{\left(\frac{\pi}{2x_\infty}\right)^3} \right] + \left[\frac{1}{\left(\frac{\pi}{2x_\infty}\right)^3} + \left(\frac{1+\left(\frac{\pi}{2x_\infty}\right)}{2\left(\frac{\pi}{2x_\infty}\right)^2 x_\infty} \right) x^2 \right] \right\} + c_2 \left\{ \left[\frac{Sin\left(\frac{\pi x}{2x_\infty}\right)}{\left(\frac{\pi}{2x_\infty}\right)^3} \right] - \left[\frac{x}{\left(\frac{\pi}{2x_\infty}\right)^2} \right] \right\} \quad (18)$$

Again, the function expressed in Eq. (18) should satisfy the boundary conditions of (15). The boundary conditions of Eq. (15a) are automatically satisfied. Thus, c_1 and c_2 are obtained using the two other boundary conditions which yields the following statement for $f(x)$

$$f(x) = (0.0668) \left\{ \left[-\frac{Cos\left(\frac{\pi x}{2x_\infty}\right)}{\left(\frac{\pi}{2x_\infty}\right)^3} \right] + \left[\frac{1}{\left(\frac{\pi}{2x_\infty}\right)^3} + \left(\frac{1+\left(\frac{\pi}{2x_\infty}\right)}{2\left(\frac{\pi}{2x_\infty}\right)^2 x_\infty} \right) x^2 \right] \right\} + (0.0559) \left\{ \left[\frac{Sin\left(\frac{\pi x}{2x_\infty}\right)}{\left(\frac{\pi}{2x_\infty}\right)^3} \right] - \left[\frac{x}{\left(\frac{\pi}{2x_\infty}\right)^2} \right] \right\} \quad (19)$$

3. Results and Discussions

An approximate integral solution is introduced for solving the well-known Blasius boundary-layer problem. The method is extended from the integral solution idea and the application of the best fitting theorem. Two types of Fourier series i.e. the Legendre's polynomial and the trigonometric expansion are employed to examine the effect of approximating of the nonlinear term on the accuracy of the solution. The present solution is compared with the numerical results obtained via the Runge-Kutta method. The results of the proposed analytical approach which are expressed in Eqs. (16) and (19) are depicted in Figure 1 in comparison with the numerical ones. As can be seen, the trigonometric approximation of the nonlinear term leads to a more accurate solution which gently follows the results of the numerical solution of f(x). On the other hand, as depicted in Figure 1, there is a considerable difference between the numerical results and the solution achieved via the Legendre's polynomial approximation.

Also, the results of the present analytical scheme as well as the numerical results of $f'(x)$ are plotted and compared in Figure 2. According to this figure, the result obtained by applying the Fourier series expansion accurately approximatesf$'(x)$ while the results achieved from the Legendre's polynomial approximation is far from the numerical solution.

Besides, the result of the $f''(x)$ obtained from the proposed solution (using trigonometric expansion) is compared with the numerical solution via Figure 3. In fluid mechanics, the value of the $f''(0)$ corresponds to the skin friction coefficient. The relative error of the $f''(0)$ between proposed analytic solution (using trigonometric approximation) and the numerical solution is reported as 16.09%.

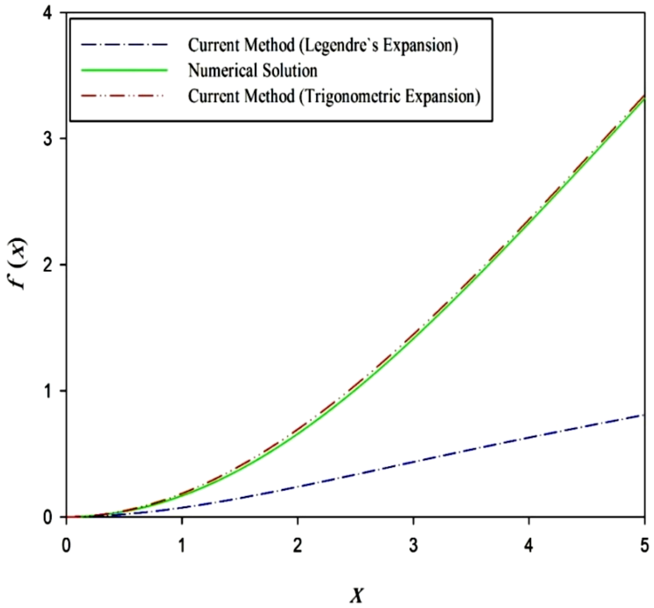

Figure 1. The results of the$f(x)$: Comparison between which are obtained from the proposed solution and that is achieved by the numerical method (using Runge-Kutta)

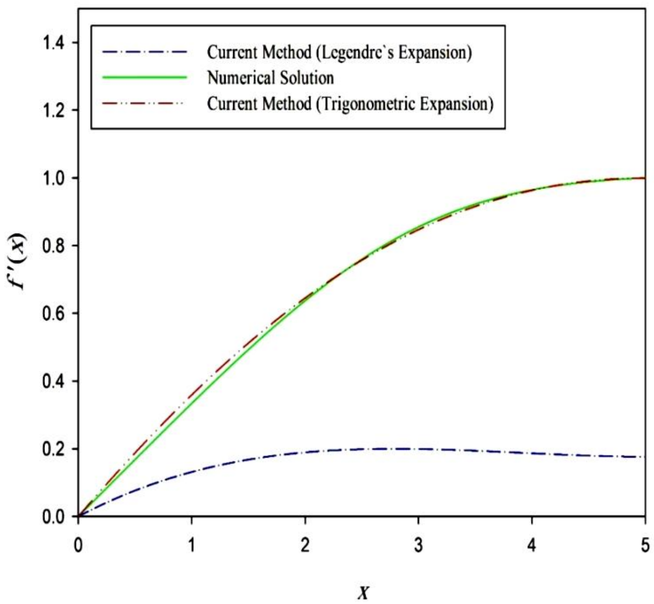

Figure 2. The results of the $f'(x)$: Comparison between which are obtained from the proposed solution and that is achieved by the numerical method (using Runge-Kutta)

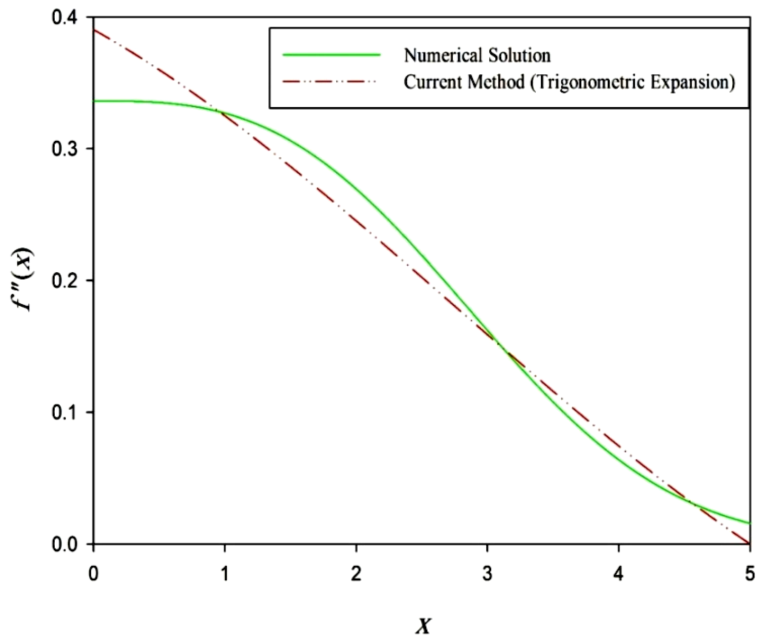

Figure 3. The results of the $f''(x)$: Comparison between which is obtained from the proposed solution (using trigonometric expansion) and that is achieved by the numerical method (using Runge-Kutta)

To gain a more sensible comparison among the mentioned techniques, Table 1 exhibits the comparison of f(x) and f′(x) obtained from the present analytic method and those via Runge-Kutta method. It is apparent from the comparison that the largest relative error belongs to $x = 0.1$ for which f(x) is about 15.90%. Besides, there is a similar relative error of 15.25% for f′(x).

Table 1. Comparison of the results obtained from the integral method (using trigonometric expansion) and the results of the Runge-Kutta method

x	$f(x)$			$f'(x)$		
	Numerical	Current method	Relative error (%)	Numerical	Current method	Relative error (%)
0	0	0	0	0	0	0
0.5	0.041983	0.047563	13.29071	0.167928	0.187714	11.78252
1	0.167579	0.184984	10.38563	0.333818	0.359057	7.560837
1.5	0.37464	0.40355	7.716779	0.492655	0.511997	3.926099
2	0.657875	0.693648	5.437783	0.637133	0.644958	1.228179
2.5	1.008163	1.044995	3.653349	0.759664	0.756854	-0.36984
3	1.413067	1.446894	2.393917	0.854978	0.847119	-0.91925
3.5	1.858528	1.8885	1.612673	0.922079	0.915717	-0.68996
4	2.331136	2.359081	1.198776	0.964407	0.96315	-0.13036
4.5	2.819988	2.848289	1.003591	0.98817	0.990436	0.229394
4.9	3.21755	3.24651	0.90015	0.99830	0.99876	0.04634
5	3.317482	3.34641	0.87198	0.999993	0.999094	-0.08992

5. Conclusions

By combining the Green's function method and the best approximation theorem, an integral solution is proposed for the well-known Blasius boundary-layer problem. A novel approximation is proposed for the nonlinear term of the Blasius problem by using a trigonometric expansion. Also, a Legendre polynomial expansion is applied to approximate this nonlinear term ($-\frac{1}{2} ff''$) and the results of the two approximations are compared with each other. In comparison with the results of numerical approach, applying the trigonometric expansion for approximating the nonlinear term, leads to a nearly accurate solution which is capable to effectively approximate f and f'. According to the accuracy of the results, it can be claimed that here an innovative integral approach is developed to approximate the solution of the Blasius equation which can be more examined for a wide range of similar flows in the boundary layer theory such as wedge flows.

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